Lecture 13— Poisson Formula

Consider $y \in B_r(z)$ we have shown in the h.w that if $g \in C(\partial B_r(z))$ then

$$u(y) = (Kz, rg)(y) := \frac{r^2 - |y-z|^2}{r|S^{n-1}|} \int\limits_{\partial B_r(z)} \frac{g(x) d^{n-1}x}{|x-y|^n}$$

is $u \in C^\infty(B_r(z))$ and solves

$$\begin{cases} \Delta u = 0 & in\ B_r(z) \\ u = g & on\ \partial B_r(z) \end{cases}$$

Conversely, if $\Delta u = 0$ in $B_r(z)$, with $u \in C^2(B_r(z)) \cap C^0(\overline{B_r(z)})$

$$\implies u = K_{z,r}(u|_{\partial B_r(z)})$$

Removable Singularity Theorem

Theorem 1. *$u \in C^2(\Omega/\{0\})$, $\Delta u = 0 \ \in \Omega/\{0\}$. We assume boundedness; that is, $u(x) = O(1)$ as $|x| \to 0$. In precise,*

$$\exists M < \infty,\ r > 0\ s.t\ |u(x)| < M\ for\ B_r/\{0\},$$

Then u can be extended to Ω as a harmonic function, i.e $u(0)$ can be chosen so that $\Delta u = 0$ in Ω.

Proof. Choose $\overline{B_R} \subset \Omega$, $\delta \in (0, R)$. Let v be such that $\begin{cases} \Delta v = 0 & \in B_r \\ v = u & on\ \partial B_R \end{cases} \implies w = u - v$. Clearly, $\Delta w = 0 \ \in B_R/B_\delta$. $w = 0$ on ∂B_R. $|v| \le M$ by Maximum principal, and so $|w| \le 2M$. Set

$$\phi(x) = \frac{2M\delta^{n-2}}{|x|^{n-2}} \qquad (\Delta\phi = 0)$$

by construction $\phi \ge 0$ on ∂B_R meanwhile $\phi = 2M$ on ∂B_δ We use the comparison principal

$$\implies w \le \phi\ on\ \partial(B_R/B_\delta) \implies w \le \phi\ in\ B_R/B_\delta.$$

$$-w \le \phi\ on\ \partial(B_R/B_\delta) \implies w \ge -\phi\ in\ B_R/B_\delta$$

$$\implies |w| \le \phi$$

hence

$$|w(x)| \le \frac{2M\delta^{n-2}}{|x|^{n-2}} \to 0\ as\ \delta \to 0 \implies w = 0\ in\ B_R/\{0\}$$

∎

Ex. $|w(x)| = o(|x|^{2-n})$.

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Harnack Inequality

Consider $B_R(y) \supset B_r(x)$ $x \in B_R(y)$ and $u \geq 0$ $\Delta u = 0$, then

$$u(x) = \frac{1}{|B_r|}\int_{B_r(x)} u \leq \frac{1}{|B_r|}\int_{B_R(y)} u = \frac{|B_R|}{|B_r|}u(y)$$

in other words the case you have two balls of x and y far from each other you can connect them by a chain of balls and reppply the thing above to get to y suppose K compact in Ω then

$$\exists C_k \ s.t \ u(x) \leq c_k u(y) \qquad \forall x, y \in K$$

Theorem 2. *Ω domain $K \subset \Omega$ compact*

$$\exists C_k \ s.t \ u(x) \leq c_k u(y) \qquad \forall x, y \in K, \ u \geq 0 \ harmonic \ in \ \Omega$$

Problem can happpen if the balls of x and y are not 'nicely' connected. ie. maybe narrow connection of hamronicity.

Proof. Define

$$S(x, y) = \sup\left\{\frac{u(x)}{u(y)} : \ u \geq \ harmonic \ in \ \Omega\right\}$$

Want : $\exists M < \infty \ s.t \ S(x, y) \leq M \qquad (x, y) \in \Omega \times \Omega$
Claim: $S(x, y) < \infty \qquad x, y \in \Omega.$

$$x \in \Omega \qquad \Sigma_x = \{y \in \Omega : S(x, y) < \infty\} \qquad x \in \Sigma_x$$

$$y \in \Sigma_x. \ \exists \overline{B_r(y)} \subset \Omega \implies u(z) \leq Cu(y) \qquad z \in B_\rho(y)$$

$$\implies S(x, z) \leq S(x, y) \implies \Sigma_x \ open.$$

Now,

$$\exists \{y_k\} \subset \Sigma_x, \ y_k \to y \in \Omega \implies \Sigma_x \ closed. \implies \quad \Sigma_x = \Omega.$$

Claim proved. Consider compact $(a, b) \in K \times K$ and define $\delta = dist(k, \partial\Omega)$.

$$x \in B_r(a), \ y \in B_r(b) \qquad \delta > 0 \ with \ r = \delta/2$$

there exists constant C_r

$$u(y) \in C_r(b), \qquad u(x) \geq \frac{1}{C_r}u(a)$$

$$(x, y) \in B_r(a) \times B_r(b)$$

$$\frac{u(y)}{u(x)} \leq \frac{C_r u(b)}{1/C_r u(a)} \leq C_r^2 S(a, b)$$

hence

$$S(x, y) \leq C_r^2(a, b) \qquad M = C_r^2 \max_{a,b} S(a, b).$$

$\square$

Harnack Thm I

Theorem 3. *Consider $\Omega \subset \mathbb{R}^n$ bdd domain and $\{u_1, u_2 ...\}$ sequence of harmonic functions in Ω, converging uniformly on the boundary $\partial\Omega$. Then,*

$$u_i \to u \qquad uniformly \qquad in\ \Omega\ with\ \Delta u = 0.$$

Furthermore, $\forall K \subset \Omega$ compact, $\forall \alpha$ multi-index,

$$\partial^\alpha u_i \to \partial^\alpha u \qquad uniformly \qquad in\ K.$$

Proof. $u_i - u_j$ is harmonic in Ω. By MP

$$|u_i - u_j| \leq \sup_{\partial\Omega} |u_i - u_j| \implies u_i \xrightarrow{unif} u\ ,\ hence\ u \in C^0(\Omega)$$

$$and$$

$$\lim_{i\to\infty} u_i(x) = \frac{1}{|B_r|} \lim_{i\to\infty} \int\limits_{B_r(x)} u_i \tag{1}$$

$$= \frac{1}{|B_r|} \int\limits_{B_r(x)} \lim_{i\to\infty} u_i = \frac{1}{|B_r|} \int\limits_{B_r(x)} u. \tag{2}$$

In more details, passing the limit under the integral is due to

$$\int\limits_{B_r} |u_i - u| \leq |B_r| \sup_{B_r(x)} |u_i - u|.$$

This concludes that the MVP holds for u so u is harmonic in Ω. In particular $u \in C^\omega(\Omega)$, hence all derivatives exist for u, and so

$$|\partial^\alpha u_i(x) - \partial^\alpha u(x)| \leq |\alpha|! \left(\frac{ne}{r}\right)^{|\alpha|} \underbrace{\sup_{B_r(x)} |u_i - u|}_{\to 0}$$

Noting that the compactness of K insures that r remains finitely away from zero, as to allow the inequality to vanish as $\sup_{B_r(x)} |u_i - u|$ vanishes as $i \to \infty$. □

Harnak II

Theorem 4. *Ω bdd domain. $u_1 \leq u_2 \leq ...$ seq of harmonic functions in Ω then either*

$$u_i(x) \to \infty \qquad \forall x \in \Omega.$$

or

$$u_k \to u \qquad loc.\ uniformly\ in\ \Omega\, with\ \Delta u = 0.$$

Proof. Suppose $u(x) \leq M < \infty$ for some M. $K \subset \Omega$ compact with $K \ni x$. By monotonicity of $\{u_i\}$ and compactness, (in particular boundedness) of K, the monotonic convergence theorem implies

$$u_i(x) \to \xi \in \mathbb{R}. \tag{3}$$

by Harnack inequality, $\exists c_k > 0$ for every k

$$\frac{1}{c_k}(u_{j+m}(x) - u_j(x)) \leq u_{j+k}(y) - u_j(y) \leq c_k(u_{j+m}(x) - u_j(x)).$$

By the previous result (3), the RHS and LHS of the inequality above defines Cauchy sequences, hence

$$u_j(y) \to u(y) \quad \textit{on } K$$

hence $u_j \to u$ locally uniformly. By Harnack I, u is harmonic. This is done by taking any compact set K and we have uniform convergence. $\square$